Approximate Method For Solving Boundary-Layer Equations

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Introduction

ONLY a few methods, (e.g., Thwaites,[1] Schlichting,[2] Head,[3] and Curle[4]) are available for determining the incompressible flow in the laminar boundary layer on a permeable surface. These methods, excluding Head's, are one-parameter and are less satisfactory for flows over permeable surfaces than for flows over solid boundaries, for the reason that the one-parameter method fails to satisfy the condition of the change of curvature of velocity profile which arises due to the presence of suction. From this point of view, the one-parameter method gives only a rough indication of the effects of suction, whereas Head's two-parameter method gives as accurate a result for the flow with suction as for the flow over an impermeable surface. Although the method of Head is more accurate than any other approximate method, it is more lengthy and more difficult to handle. So, it would be useful to find an accurate two-parameter method which is as simple as the one-parameter method of Curle. The purpose of this Note is three-fold: first, it is to present the numerical solutions of incompressible boundary-layer equations with uniform suction obtained by using the method given by Williams.[5] The numerical solutions are shown in Fig. 1 for the case of retarded flows. Second, these numerical solutions are used to extend the method of Curle to two-parameter method. Last, it is shown that for small values of suction velocity, the method of Curle is as good as the present two-parameter method because of its simplicity and accuracy.

Development of Theory for the Outer and Inner Layer

For the co-ordinate system, x and z are taken to be the axes along and perpendicular to the flow direction. Curle's method is based on Stratford's[6] basic ideas of dividing the boundary layer into the outer part near the edge of the layer and the inner part near the wall.

Only the flows in which the pressure is constant between the positions $x = 0$ and $x = x_0$ are considered, and then a sharp pressure rise, together with the contribution from suction distribution starting at $x \geqq x_0$. In the outer layer the pressure gradient is assumed to be large so that the pressure forces dominate the viscous forces. In such circumstances, the outer layer is developed by first assuming inviscid flow in the region $x \geqq x_0$, $z \geqq z_j$ so that the total head along a streamline is nearly constant. (The suffix j refers to the values where the inner and outer layers join.) Then the solution for the velocity profile is

$$(\tfrac{1}{2}\rho u^2)x, \psi = (\tfrac{1}{2}\rho u^2)x_0, \psi - (p - p_0) + \Delta H \tag{1}$$

Where the stream function, Ψ, is given by

$$\psi = \int_{x_0}^{x} w_s dx + \int_0^z u\, dx \tag{2}$$

and ΔH is the small increment due to the effect of viscosity and suction. Since it is intended here that the solution be valid for small values of w_s, the contribution to ΔH from the suction effect is assumed to be negligible. Similarly, the pressure rise being considered is of such a type that it does not greatly affect the shape of the outer profile. If this is so, the viscous forces on any fluid element are approximately the same as they would be without pressure rise or suction effect. Accordingly, the change of total head is given by

$$\Delta H = (\tfrac{1}{2}u_B^2)_{x,\psi} - (\tfrac{1}{2}u_B^2)x_0, \psi \tag{3}$$

where u_B is the Blasius flat-plate solution with no pressure gradient. Superimposing Eq. (3) into Eq. (1) gives

$$u^2(x, \psi) = u_B^2(x, \psi) - 2(p - p_0)/\rho \tag{4}$$

Since p depends only on x, we have

$$(\partial u/\partial z)x, \psi = (\partial u_B/\partial z)x, \psi \tag{5}$$

$$(\partial^2 u/\partial z^2)x, \psi = [(u/u_B)\, \partial^2 u_B/\partial z^2]x, \psi \tag{6}$$

Equations (4–6) determine the shape of the outer profile.

In the inner layer, by contrast, the suction effect and viscous force become very important so that the resultant viscous force together with the suction effect must always be comparable with the pressure gradient force. To satisfy these conditions the inner profile is taken in the form

$$\mu u = \tau z + \tfrac{1}{2}[(dp/dx) - (\tau w_s/v)]z^2 + az^n \tag{7}$$

where τ is the skin friction $\mu(\partial u/\partial z)_0$ and a and the parameter n, which depend on suction velocity, are arbitrary.

For joining the two solutions for the inner and outer regions of the boundary layer, Stratford's theory is followed, and hence it is assumed that

$$u_B(x, \psi_j)/u_0 \leqq k \tag{8}$$

where k is less than unity, to have the Blasius profile linear throughout the inner layer. From Eqs. (5) and (6), we have

$$\left(\frac{\partial u}{\partial z}\right)_j = \left(\frac{\partial u_B}{\partial z}\right)_{x,\psi_j} = \frac{\tau_B}{\mu}, \quad \left(\frac{\partial^2 u}{\partial z^2}\right)_j = \frac{u_j}{u_B(x, \psi_j)}\left(\frac{\partial^2 u_B}{\partial z^2}\right)x, \psi_j = 0$$

approximately, where τ_B is the skin friction for the Blasius layer. Similarly

$$\psi_j = \mu u_B^2(x, \psi_j)/2\tau_B$$

The inner solution for the region near the wall must be smoothly joined to the outer solution at a suitable point where ψ, u, and their derivatives are assumed to be continuous. Thus, the two solutions are joined by satisfying the conditions

$$u_j^2 = (2\tau_B/\mu)\psi_j - [2(p - p_0)/\rho] \tag{9}$$

$$\mu\psi_j = \mu \int_{x_0}^{x} w_s dx + \frac{1}{2}\tau z_j^2 + \frac{1}{6}\left(\frac{dp}{dx} - \frac{\tau w_s}{v}\right)z_j^3 + \frac{a}{n+1}z_j^{n+1} \tag{10}$$

$$\mu u_j = \tau z_j + \frac{1}{2}\left(\frac{dp}{dx} - \frac{\tau w_s}{v}\right)z_j^2 + a z_j^n \tag{11}$$

$$\tau_B = \tau + \left(\frac{dp}{dx} - \frac{\tau w_s}{v}\right)z_j + na z_j^{n-1} \tag{12}$$

The solution of these five equations leads to the following result

$$x^2\left\{\frac{dc_p}{dx} - \frac{2\tau w_s}{\mu u_0^2}\right\}^2 \left\{c_p - \frac{2\tau_B}{\mu u_0^2}\int_{x_0}^{x} w_s dx\right\} = \frac{0.00405(n-1)^3(n^2+3)}{n^2(n+1)(n-2)^2}(1-T)^3\left\{1 + \frac{3(n^2-1)}{n^2+3}T\right\} \tag{13}$$

where c_p is the pressure coefficient, defined as

$$c_p = (p - p_0)/(\tfrac{1}{2}\rho u_0^2) \tag{14}$$

and T is the nondimensional skin friction τ/τ_B.

Curle[4] takes the parameter $n = 3.043$ in his method. In this synoptic, the parameter n is treated as a function of suction velocity, and will be determined by using some known results of boundary-layer equations for different values of suction velocity. Before determining n, the form of suction function that should take must be chosen. The choice of suction function is made as follows: when the flow of fluid is over a permeable surface, it is physically acceptable that the suction effect is due to the total fluid that has been sucked away at a certain stage. This suggests $\int w_s dx$ as a useful measure of suction effect. Further, the stream function at wall, given by

$$\psi = \int_{x_0}^{x} w_s dx$$

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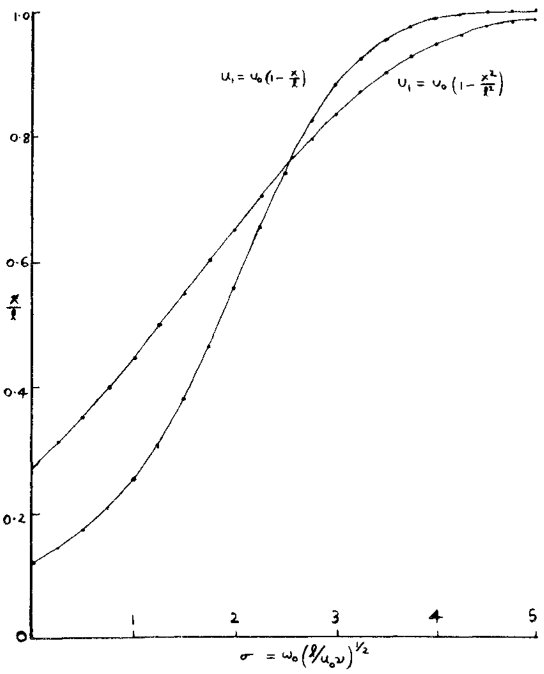

Fig. 1 Position of separation found out numerically for $u_1 = u_0[1-(x/l)^c]$ with uniform suction.

represents the displacement of streamlines. Hence the parameter n is taken to be a function of

$$G = (xu_0\nu)^{-1/2}\int_{x_0}^{x} w_s\,dx \tag{15}$$

The parameter n, which is a function of G, is determined by using the numerical solutions of boundary-layer equations with suction for retarded flow. For retarded flow, $\tau = 0$, and the general mainstream velocity is

$$u_1 = u_0[1-(x/l)^c] \tag{16}$$

where c is a positive integer. The pressure coefficient c_p for the retarded flow is

$$c_p = 1-(u_1/u_0)^2 = 2(x/l)^c-(x/l)^{2c} \tag{17}$$

Hence, Eq. (13) becomes

$$\left(\frac{x}{l}\right)^{2c}\left[1-\left(\frac{x}{l}\right)^c\right]^2\left\{2\left(\frac{x}{l}\right)^c-\left(\frac{x}{l}\right)^{2c}-\frac{0.66412}{(uvx)1/2}\int_0^x w_s\,dx\right\} = \frac{0.0010125(n-1)^3(n^2+3)}{c^2n^2(n+1)(n-2)^2} \tag{18}$$

When uniform suction w_0 is applied from the leading edge then Eq. (18) indicates that separation for the retarded flow occurs where

$$\left(\frac{x}{l}\right)^{2c}\left[1-\left(\frac{x}{l}\right)^c\right]^2\left\{2\left(\frac{x}{l}\right)^c-\left(\frac{x^2}{l^2}\right)^{2c}-0.66412\sigma\left(\frac{x}{l}\right)^{1/2}\right\} = \frac{0.0010124(n-1)^3(n^2+3)}{c^2n^2(n+1)(n-2)^2} \tag{19}$$

where

$$\sigma = w_0(l/u_0 v)^{1/2}$$

Equation (15) becomes

$$G = \sigma(x/l)^{1/2} \tag{20}$$

The boundary-layer equations are solved numerically by using the Williams's method. The numerical solutions of boundary-layer equations for the retarded flow are shown in Fig. 1.

The numerical solution for the linearly retarded flow is used to find an expression for the parameter n. Equation (19) with $c = 1$ (i.e., linearly retarded flow) is solved for n having given σ and the corresponding value of x/l. The approximate expression

Table 1 Suction required for zero skin-friction layer with linearly retarded mainstream

	$(u_0\nu l)^{-1/2}\int_0^x w_s\,dx$			$w_s(l/u_0\nu)^{1/2}$		
x/l	Head[3]	Curle[4]	Present method	Head[3]	Curle[4]	Present method
0.12	0.000	−0.004	−0.0038			
0.13				0.65	2.48	1.65
0.14	0.013	0.046	0.031			
0.15				0.88	2.25	1.61
0.16	0.030	0.091	0.067			
0.17				1.08	2.12	1.60
0.18	0.051	0.133	0.104			
0.19				1.23	2.05	1.60
0.20	0.076	0.174	0.143			

for the function $n(G)$ to fit the solutions of Eq. (19) for various values of σ and x/l is found to be

$$n = (a_0+a_2X^2+a_4X^4)/(1-b_2X^2) \tag{21}$$

where

$$X = G-b_0$$
$$a_0 = 2.7,\quad a_2 = 17.327,\quad a_4 = -14.318$$
$$b_0 = 0.402,\quad b_2 = 7.157$$

and G is given by Eq. (20). When there is no suction $X = -b_0$ and $n \simeq 3$.

Application to a Flow with Pressure Gradient

It is known that it is possible to maintain zero skin friction by applying a correct amount of suction downstream of the point $x/l = 0.121$ which is the position of separation as given by Eq. (19) for $\sigma = 0$, $c = 1$, and $n \simeq 3$.

The suction quantity

$$(u_0\nu l)^{-1/2}\int_0^x w_s\,dx$$

required to preserve zero skin-friction layer, is obtained as a function of x/l from Eq. (19) with $c = 1$ and then the suction velocity, equal to $w_s[l/(u_0\nu)]^{1/2}$, by differentiation. The results obtained this way for suction velocity and suction quantity to preserve the zero skin-friction layer for the linearly retarded flow are given in Table 1, together with the results obtained by Head and Curle. It can be seen from Table 1 that the present method gives more accurate results than that of Curle.

Application to a Flow with Zero Pressure Gradient

In this section, the flow past a flat plate with uniform suction will be considered. Since the mainstream velocity u_1 is constant for the flow past a flat plate, the pressure gradient c_p is zero and hence Eqs. (13) and (15) reduce to

$$w_s\left(\frac{x}{u_0\nu}\right)^{1/2}\left\{\frac{1}{w_s x}\int_0^x w_s\,dx\right\}^{1/3} = F(n)$$

Table 2 Skin friction for flat plate with uniform suction

ξ (Floating decimal)	$\xi^{1/2}$ (Floating decimal)	$T = \tau/\tau_B$ Exact Iglisch	Curle	Present method
2.07024, −3	4.5500, −2	1.0901	1.1	1.09743
7.69129, −3	8.7700, −2	1.1602	1.2	1.1917
2.69945, −2	1.6430, −1	1.3039	1.4	1.3714
5.43822, −2	2.3320, −1	1.4602	1.6	1.5428
8.77344, −2	2.9620, −1	1.5920	1.8	1.7095
1.26025, −1	3.5500, −1	1.7202	2.0	1.8769
2.37754, −1	4.8760, −1	1.0905	2.5	2.3165

where

$$F(n) = \left[\frac{0.0138(n-1)^3(n^2+3)}{n^2(n+1)(n-2)^2}\right]^{1/3} \times (T-1)\left[\frac{1}{T^2}\left\{1+\frac{3(n^2-1)T}{(n^2+3)}\right\}\right]^{1/3} \quad (22)$$

When there is uniform suction downstream of the leading edge of flat plate, then Eq. (22) becomes

$$\xi^{1/2} = F(n) \quad (23)$$

where

$$\xi = w_0{}^2 x/u_0 \nu \quad (24)$$

The result calculated from Eq. (23) is given in Table 2 together with Iglisch exact solution and Curle approximate solution for the purpose of comparison. The comparison in Table 2 shows that the present approximate method produces results which are slightly more accurate than those given by Curle.

References

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On Statistical Analysis of Composite Solid Propellant Combustion

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Hermance[1] and Beckstead, Derr, and Price[2] have formulated steady-state combustion models that employ statistical concepts. This is an important advance because it leads to an "explicit" and calculatable relation between oxidizer particle size and burning rate. The objectives of this Note are first to show that the statistical portions of these analyses, which are virtually identical, are questionable and second to reformulate the problem "correctly" for the assumed burning surface topography. The reformulation is a generalization of the pioneering work of Miller, Hartman, and Myers.[3]

The burning surface of a composite solid propellant is an ensemble (like a patch work quilt) of different flamelets.† In Refs. 1 and 2 the ensemble is replaced with a single flamelet and the burning surface topography is assumed to be a fuel plane dotted with concave and/or convex oxidizer surfaces. The characteristic spatial dimension(s) of the single flamelet is based on the mean oxidizer particle intersection area of a planar "cut" through a unimodal propellant with randomly packed spherical oxidizer particles. In such a "cut" the oxidizer particle intersection areas range from zero to $\pi D_o{}^2/4$‡ and the mean intersection area is $\pi D_o{}^2/6$. It is important to note that the characteristic dimension is based solely on geometry.

The Hermance and Beckstead, Derr, and Price approaches are questionable on the following two points: 1) the physical validity of replacing the behavior of an ensemble of different flames with that of a single flame (a basic assumption in most quasi-one-dimensional analyses of composite solid propellant combustion), and 2) the statistical process employed to select the characteristic dimension. Both points are related. The total oxidizer vapor flow from the burning surface M_{ox} is the sum of the oxidizer flows from all of the individual oxidizer surfaces so

$$M_{ox} = \sum_i m_{ox,i}\,\varepsilon_{ox,i} \quad (1)$$

where $m_{ox,i}$ is the flux of oxidizer from and $\varepsilon_{ox,i}$ the area of the ith oxidizer surface. For quasi-steady conditions the mixture ratio is fixed by the composition of the solid. Therefore

$$\bar{m}_T = \sum_i m_{ox,i}\,\varepsilon_{ox,i}/(\alpha S_o) \quad (2)$$

Numerical results from both Refs. 1 and 2 show that, in general, $m_{ox,i}$ is a function of $\varepsilon_{ox,i}$. Consequently, a characteristic dimension selected from geometry alone is not compatible with Eq. (2) so that the statistical aspects of Refs. 1 and 2 appear to violate continuity. It should be noted that criticism is not directed at the physiochemical combustion models.

The reason for difficulty is clear. The statistical aspects must be compatible with Eq. (2). Let ε_f and ε_{ox} be the fuel and oxidizer surfaces associated with a flamelet and F_{ox} and F_f be distribution functions such that the fraction of oxidizer surfaces with $\varepsilon_{ox} \leqq \varepsilon_{ox} \leqq \varepsilon_{ox} + d\varepsilon_{ox}$ is $F_{ox}\,d\varepsilon_{ox}$ and the fraction of fuel surfaces with $\varepsilon_f \leqq \varepsilon_f \leqq \varepsilon_f + d\varepsilon_f$ is $F_f\,d\varepsilon_f$. Then, if N is the number of oxidizer surfaces per unit planar area of burning surface, the number of flames with $\varepsilon_{ox} \leqq \varepsilon_{ox} \leqq \varepsilon_{ox} + d\varepsilon_{ox}$ is

$$dN = NF_{ox}\,d\varepsilon_{ox} \quad (3)$$

Since the solid propellant is a random packing, assume that the distribution functions are independent.§ Consequently, the fraction of the dN flamelets with $\varepsilon_f \leqq \varepsilon_f \leqq \varepsilon_f + d\varepsilon_f$ is

$$d^2N = NF_f F_{ox}\,d\varepsilon_f \quad (4)$$

If it is further assumed that the individual flamelets are non-interacting (each flamelet does its own thing), the mass flux of oxidizer from each oxidizer surface becomes a function of pressure, intial propellant temperature, freestream velocity, oxidizer and fuel surface area, etc. The mass flux of oxidizer from flamelets with $\varepsilon_{ox} \leqq \varepsilon_{ox} \leqq \varepsilon_{ox} + d\varepsilon_{ox}$ and $\varepsilon_f \leqq \varepsilon_f \leqq \varepsilon_f + d\varepsilon_f$ is

$$d^2\bar{m}_{ox} = Nm_{ox}\varepsilon_{ox}F_{ox}F_f\,d\varepsilon_{ox}\,d\varepsilon_f \quad (5)$$

The mass flow of oxidizer from flamelets with $\varepsilon_{ox} \leqq \varepsilon_{ox} \leqq \varepsilon_{ox} + d\varepsilon_{ox}$ is obtained by "summing" over all possible ε_f so

$$d\bar{m}_{ox} = NF_{ox}\varepsilon_{ox}\left(\int_{\varepsilon'_f}^{\infty} m_{ox}F_f\,d\varepsilon_f\right)d\varepsilon_{ox} \quad (6)$$

The infinite upper limit occurs because there is no specific physical upper bound to ε_f. Of course, the probability of large ε_f is vanishingly small.¶ The finite lower limit occurs because oxidizer particles cannot interpenetrate. The mean mass flux of

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† Flamelet is employed to denote the complex multi-flame structure associated with specific oxidizer/fuel surface pairs.

‡ Nomenclature is the same as that of Ref. 2.

§ Although this assumption seems reasonable, further study is warranted. It is a complex question because detailed knowledge of the statistical nature of the random packing is required.

¶ Note the similarity here to the kinetic theory of gases where each velocity component ranges from $-\infty$ to ∞.